

Basic Counting: Permutations and Combinations

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1. Introduction: A Real-World Counting Problem

Teaching Strategy Festival

Schedule

Set 1 - 30 min.

Intermission

Set 2 - 30 min.

Encore - 25 min.

Blurred image: a concert photo, part of a flyer for a Teaching Strategy Festival.

Teaching Strategy Festival © Dr. Shirley L. Yu

1. Introduction: A Real-World Counting Problem

Setting

- 4 singers, 4 guitarists, 4 drummers, and 4 bassists
- Set 1 – form 4 bands and perform
- Set 2 – form 4 bands and perform
- (band = singer + guitarist + drummer + bassist)

Question

- If the musicians form bands in a completely random manner, what are the odds that in Set 2, none of them performs with any of their former bandmates in Set 1?
- Vote for your best guess (options provided) on Padlet!

1. Introduction: A Real-World Counting Problem

Answer

- ① In Set 2, there are 13 824 ways to form 4 bands.
- ② In 24 of these ways, none of the musicians performs with any of their former bandmates in Set 1.
- ③ $\Rightarrow$ The odds are $24/13\,824 \approx 0.17\%$.

Big Picture

- **Counting** is the process of determining the number of elements of a finite set of objects.
- **Probability** is a branch of mathematics and statistics concerning events and numerical descriptions of how likely they are to occur.

1. Introduction: Learning Outcomes

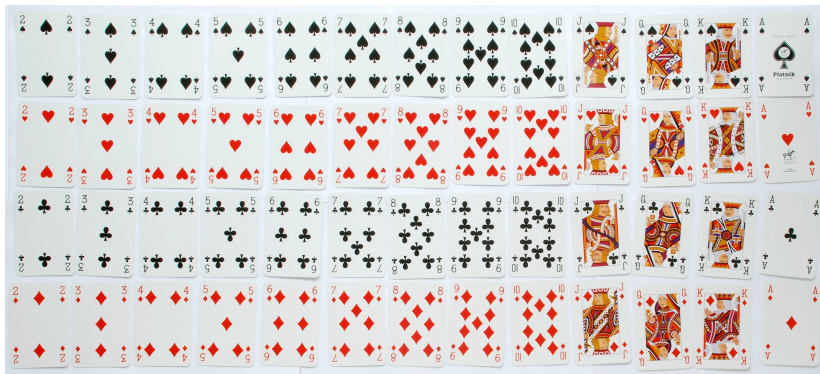
Students Will Be Able To

- Define permutations and use factorials to count them.
- Define combinations and use binomial coefficients to count them.
- Formulate real-world (e.g., poker) counting problems in mathematical language.
- Solve simple counting problems and incorporate ordering when appropriate.
- (Optional) Appreciate the usefulness of mathematics in daily life.

2. Permutations and Factorials

Standard 52-Card Deck

- 4 suits: spades, hearts, clubs, and diamonds.
- 13 values: 2 to 10, jack, queen, king, and ace.



Piatnik English pattern pack. © Trainler, CC BY 3.0



2. Permutations and Factorials

Permutation Example 1

- 5 cards: 5♣, 6♦, 7♥, 7♠, and J♣.
- How many ways are there to line them up?

Answer: One Card at a Time

- ① There are 5 ways to choose the first card.
- ② Then there are 4 ways to choose the second card.
- ③ Then there are 3 ways to choose the third card.
- ④ Then there are 2 ways to choose the fourth card.
- ⑤ Then there is only 1 way to choose the fifth card.
- ⑥ $\Rightarrow$ There are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways in total.

2. Permutations and Factorials

Permutation Example 2

- 5 cards: 5♣, 6♦, 7♥, 7♠, and J♣.
- How many ways are there to pick 3 of these 5 cards and line them up?

Answer: One Card at a Time

- ① There are 5 ways to choose the first card.
- ② Then there are 4 ways to choose the second card.
- ③ Then there are 3 ways to choose the third card.
- ④ $\Rightarrow$ There are $5 \times 4 \times 3 = 60$ ways in total.

2. Permutations and Factorials

Is there a notation to simplify these?

- Permutation Example 1: $5 \times 4 \times 3 \times 2 \times 1 = 120$.
- Permutation Example 2: $5 \times 4 \times 3 = 60$.

Yes! We have the factorial function!

If n is a positive integer, then $n! = n \times (n - 1) \times \cdots \times 1$.

Then our answers can be simplified to:

- Permutation Example 1: $5! = 120$.
- Permutation Example 2: $5!/2! = 60$.

2. Permutations and Factorials

Definition and General Formulae

- Permutation: Way to order a set of objects.
- There are $n!$ ways to order a set of n objects.
- There are $n!/(n - k)!$ ways to select and order k objects from a set of n .

Wait! Do you really understand factorials?

- Activity: Which two factorials is the number of seconds in a century between?
- Vote for your best guess (options provided) on Padlet!
- Answer: $12! = 479\,001\,600$, number of seconds in a century $= 3\,155\,695\,200$, $13! = 6\,227\,020\,800$.

3. Combinations and Binomial Coefficients

From Permutations to Combinations

- 5 cards: 5♣, 6♦, 7♥, 7♠, and J♣.
- How many ways are there to pick 3 of these 5 cards if order does not matter?

How many all-black permutations are there?

5♣, 6♦, 7♥	5♣, J♣, 7♥	6♦, 7♠, J♣	7♥, 7♠, 5♣	7♠, 6♦, 7♥	J♣, 5♣, 7♠
5♣, 6♦, 7♠	5♣, J♣, 7♠	6♦, J♣, 5♣	7♥, 7♠, 6♦	7♠, 6♦, J♣	J♣, 6♦, 5♣
5♣, 6♦, J♣	6♦, 5♣, 7♥	6♦, J♣, 7♥	7♥, 7♠, J♣	7♠, 7♥, 5♣	J♣, 6♦, 7♥
5♣, 7♥, 6♦	6♦, 5♣, 7♠	6♦, J♣, 7♠	7♥, J♣, 5♣	7♠, 7♥, 6♦	J♣, 6♦, 7♠
5♣, 7♥, 7♠	6♦, 5♣, J♣	7♥, 5♣, 6♦	7♥, J♣, 6♦	7♠, 7♥, J♣	J♣, 7♥, 5♣
5♣, 7♥, J♣	6♦, 7♥, 5♣	7♥, 5♣, 7♠	7♥, J♣, 7♠	7♠, J♣, 5♣	J♣, 7♥, 6♦
5♣, 7♠, 6♦	6♦, 7♥, 7♠	7♥, 5♣, J♣	7♠, 5♣, 6♦	7♠, J♣, 6♦	J♣, 7♥, 7♠
5♣, 7♠, 7♥	6♦, 7♥, J♣	7♥, 6♦, 5♣	7♠, 5♣, 7♥	7♠, J♣, 7♥	J♣, 7♥, 6♦
5♣, 7♠, J♣	6♦, 7♠, 5♣	7♥, 6♦, 7♠	7♠, 5♣, J♣	J♣, 5♣, 6♦	J♣, 7♠, 6♦
5♣, J♣, 6♦	6♦, 7♠, 7♥	7♥, 6♦, J♣	7♠, 6♦, 5♣	J♣, 5♣, 7♥	J♣, 7♠, 7♥

3. Combinations and Binomial Coefficients

From Permutations to Combinations

- 5 cards: 5♣, 6♦, 7♥, 7♠, and J♣.
- How many ways are there to pick 3 of these 5 cards if order does not matter?

Answer: Removal of Ordering

- ① The number of permutations is $5!/2! = 60$.
- ② There are 6 all-black permutations (e.g., 5♣, 7♠, J♣).
- ③ There are $3!$ permutations for each subset of 3 cards.
- ④ If order does not matter, there are $60/3! = 10$ ways.
- ⑤ Implication: Ordering can be removed via division!

3. Combinations and Binomial Coefficients

Definitions and General Formula

- Combination: Way to choose a subset of objects if order does not matter.
- Binomial coefficient: Number of ways to choose a subset of k from n objects.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Activity: Back to the Standard 52-Card Deck!

- How many ways are there to choose 5 cards from it?
- Give your answer (formula and/or number) on Padlet!
- Answer: $\binom{52}{5} = 52!/(5! \times 47!) = 2\,598\,960$.

4. Real-World Counting Problem: Poker

Blurred image: a chart of the standard poker hands, from Poker.org.

Standard Poker Hands © Poker.org

4. Real-World Counting Problem: Poker

Let's count the (One) Pair combinations!

How many ways are there to choose 5 cards from a standard 52-card deck, so that 2 of them have the same value and the other three all have different values?

First Approach: One Card at a Time

- ① There are 52 ways to choose the first card.
- ② Then there are 3 ways to choose the second card.
- ③ There are 48 ways to choose the third card.
- ④ Then there are 44 ways to choose the fourth card.
- ⑤ Then there are 40 ways to choose the fifth card.
- ⑥ $\Rightarrow$ Answer: $52 \times 3 \times 48 \times 44 \times 40 = 13\,178\,880$.

4. Real-World Counting Problem: Poker

Second Approach: Using Binomial Coefficients

- ① Values: There are $\binom{13}{4}$ ways to choose 4 different values.
- ② Values: Then there are $\binom{4}{1}$ ways to choose the special value of which we have a pair.
- ③ Suits: For the pair, there are $\binom{4}{2}$ ways to choose 2 different suits.
- ④ Suits: For each of the “single” cards, there are $\binom{4}{1}$ ways to choose a suit.
- ⑤ $\Rightarrow$ Answer: $\binom{13}{4} \times \binom{4}{1} \times \binom{4}{2} \times \binom{4}{1}^3 = 1\,098\,240.$

4. Real-World Counting Problem: Poker

Group Activity: Instructions

- Team up with your neighbors.
- Each team is assigned a standard hand.
- Describe your chosen standard hand in mathematical language, just like we did for (One) Pair.
- Count the corresponding combinations. Write the formula and explain it. If time permits, compute it.
- Share your answer as comment(s) on the corresponding Padlet post.

4. Real-World Counting Problem: Poker

Group Activity: Answer Key

- 3. Four-of-a-Kind: $\binom{13}{2} \times \binom{2}{1} \times \binom{4}{4} \times \binom{4}{1} = \boxed{624}$
- 4. Full House: $\binom{13}{2} \times \binom{2}{1} \times \binom{4}{3} \times \binom{4}{2} = \boxed{3\,744}$
- 5. Flush + 2. Straight Flush + 1. Royal Flush:
 $\binom{13}{5} \times \binom{4}{1} = \boxed{5\,148}$ (Most of them are just Flush.)
- 6. Straight: $10 \times \left[\binom{4}{1}^5 - \binom{4}{1} \right] = \boxed{10\,200}$
- 7. Three-of-a-Kind: $\binom{13}{3} \times \binom{3}{1} \times \binom{4}{3} \times \binom{4}{1}^2 = \boxed{54\,912}$
- 8. Two Pair: $\binom{13}{3} \times \binom{3}{2} \times \binom{4}{2}^2 \times \binom{4}{1} = \boxed{123\,552}$
- 9. (One) Pair: $\binom{13}{4} \times \binom{4}{1} \times \binom{4}{2} \times \binom{4}{1}^3 = \boxed{1\,098\,240}$
- 10. High Card: $\left[\binom{13}{5} - 10 \right] \times \left[\binom{4}{1}^5 - \binom{4}{1} \right] = \boxed{1\,302\,540}$

5. Conclusion: Learning Outcomes

(Hopefully) Students Are Now Able To

- Define permutations and use factorials to count them.
- Define combinations and use binomial coefficients to count them.
- Formulate real-world (e.g., poker) counting problems in mathematical language.
- Solve simple counting problems and incorporate ordering when appropriate.
- (Optional) Appreciate the usefulness of mathematics in daily life.